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LETTER TO THE EDITOR

Invariants, characteristics and global geometry of large-*n* renormalisation group trajectories

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Abstract. A procedure is described for constructing invariants, characteristics, and non-linear scaling fields for large-*n* differential renormalisation group (RG) equations. Explicit solutions are derived for the Ginsburg-Landau-Wilson and time-dependent Ginsburg-Landau models. The characteristics facilitate a natural geometric representation of the RG trajectories which, for the static case, takes the form of two-dimensional solution surfaces.

Non-linear scaling fields have been shown by Nicoll *et al* (1974, 1975, 1976) to be a convenient representation of solutions of renormalisation group (RG) equations and to facilitate trajectory integrals that incorporate crossover effects of competing fixed points. Furthermore, non-linear scaling fields may be used to construct RG invariants to label individual trajectories, thereby producing a quantitative measure of parameter space topology, for example, by identifying separatrices and singular points.

The large-*n* limit of the classical *n*-vector model (Stanley 1968) is one case where these constructions can be carried out exactly either within the finite-difference recursive development of the Wilson RG (Ma 1973, Zannetti and Di Castro 1977) or more conveniently within a differential formulation (Wegner and Houghton 1973, Nicoll *et al* 1976, Busiello *et al* 1981, 1983a, b, Vvedensky 1984a). In fact the differential RG (DRG) approach (Wilson and Kogut 1974, Wegner and Houghton 1973, Nicoll and Chang 1977, Vvedensky *et al* 1983) provides a framework for constructing general solutions of large-*n* RG equations (Vvedensky 1984b) simply by using established techniques for partial differential equations (Courant and Hilbert 1962).

In a recent letter, Busiello *et al* (1986) developed a procedure for systematically constructing RG invariants by writing the large-*n* DRG equations as a Hamilton-Jacobi equation and then invoking standard methods of classical mechanics. Here, we show (1) that invariants of large-*n* DRG equations even as complex as those for critical dynamics may be constructed by a straightforward application of the method of characteristics (Courant and Hilbert 1962), and (2) that the characteristics facilitate a natural non-perturbative geometric representation of RG trajectories, which we illustrate explicitly for the static case. As an addendum to our earlier work (Vvedensky 1984b), we represent the general solutions of large-*n* DRG equations as generating functions for non-linear scaling fields, so demonstrating the explicit relationship between RG invariants and non-linear scaling fields for the spherical model.

We begin as usual with an isotropic d-dimensional system ($d<2$) characterised by an n-component order parameter $\psi_i(x)$, $i=1,\ldots,n$ and with the reduced Ginsburg-Landau-Wilson Hamiltonian appropriate for the limit $n\to\infty$ (Ma 1976):

$$\mathcal{H}=\int \mathrm{d}\boldsymbol{x}\,[(\nabla\boldsymbol{\psi})^2+H(\boldsymbol{\psi}^2)] \qquad (1)$$

where

$$H(\psi^2)=\sum_{p=1}^{\infty} u_{2p}(\boldsymbol{\psi}\cdot\boldsymbol{\psi})^p. \qquad (2)$$

Defining $H'(\boldsymbol{\psi}\cdot\boldsymbol{\psi})=\mathrm{d}H/\mathrm{d}\boldsymbol{\psi}^2$ and $x=\boldsymbol{\psi}^2/N$, where $N=nS_d/2(d-2)(2\pi)^d$ and S_d is the surface area of a unit d-sphere, the large-n DRG equation is given in terms of $t(x)=H'(xN)$ by (Busiello *et al* 1981, Vvedensky 1984a)

$$\frac{\partial t}{\partial l}=2t+(2-d)\left(x-\frac{1}{1+t}\right)\frac{\partial t}{\partial x} \qquad (3)$$

with the initial condition

$$t(x,0)=\sum_{p=1}^{\infty} pu_{2p}(0)(xN)^{p-1}. \qquad (4)$$

We may regard x as a function of t and l and rewrite (3) as

$$\frac{\partial x}{\partial l}=(d-2)x-2t\frac{\partial x}{\partial t}+\frac{2-d}{1+t}. \qquad (5)$$

Linearising about the fixed points we find that (3) and (5) are diagonal at the trivial and spherical fixed points, respectively.

The general solution to (3) and (5) may be derived by the method of characteristics (Courant and Hilbert 1962). The characteristics of (3) are the two-parameter family of space curves satisfying the equations

$$\mathrm{d}x/\mathrm{d}s=P \qquad \mathrm{d}t/\mathrm{d}s=Q \qquad \mathrm{d}l/\mathrm{d}s=1 \qquad (6)$$

where $P=(d-2)(x-1/(1+t))$ and $Q=2t$. The two independent solutions of the subsidiary system

$$\mathrm{d}x/P=\mathrm{d}t/Q=\mathrm{d}l \qquad (7)$$

yield the invariants (integrals of motion) of (3):

$$\xi_1=t\,\mathrm{e}^{-2l} \qquad (8a)$$

$$\xi_2=2t^{(2-d)/2}\left(x-1-\tfrac{1}{2}(d-2)t\int_0^1\frac{z^{(2-d)/2}\,\mathrm{d}z}{1+zt}\right). \qquad (8b)$$

Although the solutions (8b) represent a one-parameter family of fixed points of (3) (Busiello *et al* 1983b), the only *stable* solution in $2<d<4$ dimensions is the Ma (1973) solution, $\xi_2=0$.

The characteristics are determined by the family of space curves resulting from the intersections of the planes (8). For once-differentiable but otherwise arbitrary functions

$\mathscr{F}(x, y)$ and $\mathscr{G}_\alpha(z)$, $\alpha = 1, 2$, the general solution to (3) is thereby obtained as $\mathscr{F}(\xi_1, \xi_2) = 0$, or equivalently as either $\xi_1 = \mathscr{G}_1(\xi_2)$ or $\xi_2 = \mathscr{G}_2(\xi_1)$:

$$t = \mathrm{e}^{2l}\mathscr{G}_1\left(2t^{(2-d)/2}(x-1) - (d-2)\int_0^t \frac{z^{(2-d)/2}\,\mathrm{d}z}{1+z}\right) \tag{9a}$$

$$x = 1 + \tfrac{1}{2}(d-2)t\int_0^1 \frac{z^{(2-d)/2}\,\mathrm{d}z}{1+zt} + \tfrac{1}{2}t^{(d-2)/2}\mathscr{G}_2(t\,\mathrm{e}^{-2l}) \tag{9b}$$

where in view of (4) $\mathscr{G}_\alpha = \Sigma a_n^\alpha z^n$, where the a_n^α are constants.

There is an alternative form of the general solutions (9) that has some attractive features. Introducing $\mathscr{F}(\xi_1, \xi_2/\xi_1^{(2-d)/2}) = 0$, we obtain

$$t = \mathrm{e}^{2l}\mathscr{G}_1\left[2\,\mathrm{e}^{(2-d)l}\left(x - 1 - \tfrac{1}{2}(d-2)t\int_0^1 \frac{z^{(2-d)/2}\,\mathrm{d}z}{1+zt}\right)\right] \tag{10a}$$

$$x = 1 + \tfrac{1}{2}(d-2)t\int_0^1 \frac{z^{(2-d)/2}\,\mathrm{d}z}{1+zt} + \tfrac{1}{2}\,\mathrm{e}^{(d-2)l}\mathscr{G}_2(t\,\mathrm{e}^{-2l}) \tag{10b}$$

which produce the generating functions for non-linear scaling fields (Ma 1974, Zannetti and Di Castro 1977) in the (*a*) trivial and (*b*) spherical fixed point representations. We also see from (10*b*) that a non-zero initial function $t(x, 0) = t_0(x)$ evolves toward the spherical fixed point provided that $t_0(1) = 0$, a condition that is preserved along the RG trajectory and thus defines the critical surface.

The characteristics facilitate a natural geometric representation of the solution surface of quasi-linear partial differential equations which we illustrate here for the DRG equation (3). An important simplifying feature of the integrals of motion (8) is that $\xi_1 = \xi_1(t, l)$ and $\xi_2 = \xi_2(x, t)$, so that planes of constant ξ_1 and ξ_2 appear as lines when projected onto the t-l and x-t planes, respectively (figure 1). We may in fact deduce the qualitative behaviour of large-n RG trajectories by focusing upon ξ_2 and using ξ_1 only to indicate the direction and relative magnitude of flow along the characteristics.

The analytic properties of the large-n RG are reflected in several features of the characteristics (figure 1(*b*)).

(i) The planes of constant ξ_2 intersect along the line $x = 1$, $t = 0$, which is indicative of the singularity in (3) along this line. As discussed above the line $x = 1$, $t = 0$ determines the critical surface in the parameter space $\{u_{2p}\}$.

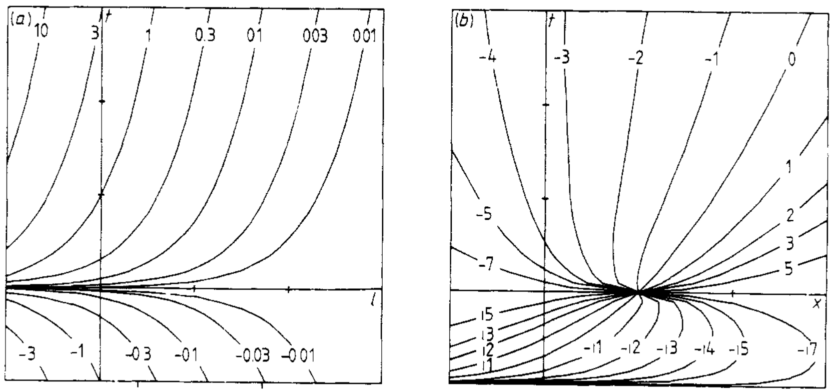

Figure 1. The invariants (8*a*, *b*) plotted in (*a*) t-l plane and (*b*) the x-t plane for $d = 3$. The values of the contours are as indicated.

(ii) For $t>0$ ξ_2 is real, while for $t<0$ ξ_2 is imaginary. Nevertheless, there are no formal difficulties in considering the range of t to be $-1<t<\infty$, despite (iii)

(iii) The singularity of (3) on the plane $t=-1$, where the equations (8) have no solution. As a glance at figure 1(b) shows, an initial function t_0 defined for $-1<t<\infty$ remains in that region for all l. Indeed, the regions $t>0$, $-1<t<0$, and $t=0$ are RG-invariant subspaces.

(iv) Although the domain of attraction of the spherical fixed point includes *all* functions defined in $-1<t<\infty$ for which $t_0(1)=0$, the analytic properties are not necessarily preserved along the RG trajectory. For example, there may appear intermediate functions at some $l'>l$ for which the quantity $t'(1, l')$ becomes unbounded. We can avoid this difficulty by confining ourselves to initial distributions for which $0<t_0'(1)<\infty$. This point was noted by Ma (1973) but not discussed in any detail.

For the time-dependent Ginsburg-Landau (TDGL) model we begin with the generalised Langevin equation

$$\dot{\psi}_i = -\Gamma(\boldsymbol{x})\frac{\delta\mathscr{H}}{\delta\psi_i(\boldsymbol{x}t)} + \eta_i(\boldsymbol{x}t) \tag{11}$$

where $\mathscr{H}$ is given by (1) and (2), Γ is a transport coefficient, and the n-component Gaussian stochastic term η is specified by

$$\langle\eta_i(\boldsymbol{x}t)\rangle = 0 \qquad \langle\eta_i(\boldsymbol{x}t)\eta_j(\boldsymbol{x}'t')\rangle = 2\Gamma(\boldsymbol{x})\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t-t'). \tag{12}$$

Introducing the field $\boldsymbol{\varphi}$ canonically conjugate to the $\boldsymbol{\eta}$, the action for the dynamics (11) is given by (Chang *et al* 1978, Szépfalusy and Tél 1980a)

$$\mathscr{A} = \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}t\{\mathrm{i}\boldsymbol{\phi}\cdot(\nabla^2\boldsymbol{\psi}+\Gamma^{-1}\dot{\boldsymbol{\psi}}) + A(\boldsymbol{\psi}\cdot\boldsymbol{\psi}, \mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi})\} \tag{13}$$

where

$$A = \sum_{p=1}^{\infty} u_{2p}(\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi})(\boldsymbol{\psi}\cdot\boldsymbol{\psi})^{p-1}. \tag{14}$$

Defining

$$x = \frac{1}{N}\boldsymbol{\psi}\cdot\boldsymbol{\psi} \qquad y = \frac{1}{2N}\frac{d+z-2}{d-2}\mathrm{i}\boldsymbol{\phi}\cdot\boldsymbol{\psi} \tag{15}$$

where the characteristic time exponent z takes on the mean-field value 4 (resp., 2) if the order parameter is conserved (resp., not conserved), the large-n DRG equations for the TDGL model are given in terms of the quantities $t_1 = \partial A/\partial x$ and $t_2 = \partial A/\partial y$ by (Busiello *et al* 1983a, Vvedensky 1984a)

$$\frac{\partial t_i}{\partial l} = \lambda_i t_i + \mu_1(x - F(t_1, t_2))\frac{\partial t_i}{\partial x} + \mu_2(y + G(t_1, t_2))\frac{\partial t_i}{\partial y} \tag{16}$$

for $i=1, 2$, where $\lambda_1 = 2+z$, $\lambda_2 = 2$, $\mu_1 = 2-d$, and $\mu_2 = 2-d-z$, and where

$$F = [(1+t_2)^2 - 2t_1]^{-1/2} \qquad G = 1-(1+t_2)F. \tag{17}$$

From (14) we see that $t_1(x, 0, l) = 0$, so that if $y=0$, the system (16) reduces to the static DRG (3). The equations (16) are written in a representation that is diagonal about the trivial fixed point. We may of course write (16) in the spherical fixed point representation:

$$\frac{\partial x}{\partial l} = \mu_1(x - F) - \lambda_1 t_1 \frac{\partial x}{\partial t_1} - \lambda_2 t_2 \frac{\partial x}{\partial t_2}$$
$$\frac{\partial y}{\partial l} = \mu_2(y + G) - \lambda_1 t_1 \frac{\partial y}{\partial t_1} - \lambda_2 t_2 \frac{\partial y}{\partial t_2}. \tag{18}$$

The four integrals of motion of (16) are given by

$$\xi_1 = \mathrm{e}^{-\lambda_1 l} t_1 \qquad \xi_2 = \mathrm{e}^{-\lambda_2 l} t_2$$
$$\xi_3 = \lambda_1 x t_1^{\mu_1/\lambda_1} + f(t_1, t_2) \tag{19}$$
$$\xi_4 = \lambda_2 y t_2^{\mu_2/\lambda_2} + g(t_1, t_2)$$

where f and g are solutions of

$$\lambda_1 t_1 \frac{\partial f}{\partial t_1} + \lambda_2 t_2 \frac{\partial f}{\partial t_2} = -\lambda_1 \mu_1 t_1^{\mu_1/\lambda_1} F$$
$$\lambda_1 t_1 \frac{\partial g}{\partial t_1} + \lambda_2 t_2 \frac{\partial g}{\partial t_2} = \lambda_2 \mu_2 t_2^{\mu_2/\lambda_2} G \tag{20}$$

with the boundary conditions $f(0, 0) = 0$ and $g(0, t_2) = 0$. The solutions to (20) may be written in the form

$$f(t_1, t_2) = \lambda_1 \mu_1 t_1^{\mu_1/\lambda_1} \int_0^1 \mathrm{d}s\, s^{\mu_1 - 1}[1 - F(s^{\lambda_1} t_1, s^{\lambda_2} t_2)]$$
$$g(t_1, t_2) = \lambda_2 \mu_2 t_2^{\mu_2/\lambda_2} \int_0^1 \mathrm{d}s\, s^{\mu_2 - 1}[G(s^{\lambda_1} t_1, s^{\lambda_2} t_2) - 1]. \tag{21}$$

The invariants (19) with $\xi_3 = \xi_4 = 0$ are thus seen to be the non-trivial fixed-point solutions derived by Szépfalusy and Tél (1980b).

We again define once-differentiable functions $\mathscr{F}_\alpha(x_1, x_2)$ and $\mathscr{G}_\alpha(x_1, x_2)$, $\alpha = 1, 2$ and introduce the quantities

$$\eta_3 = \mathrm{e}^{\mu_1 l} t_1^{-\mu_1/\lambda_1} \xi_3 \qquad \eta_4 = \mathrm{e}^{\mu_2 l} t_2^{-\mu_2/\lambda_2} \xi_4. \tag{22}$$

The general solutions to (16) may thus be represented as generating functions for non-linear scaling fields in either the trivial

$$t_i = \mathrm{e}^{\lambda_i l} \mathscr{F}_i(\eta_3, \eta_4) \tag{23}$$

or spherical fixed-point representation:

$$\eta_3 = \mathscr{G}_1(\xi_1, \xi_2) \qquad \eta_4 = \mathscr{G}_2(\xi_1, \xi_2). \tag{24}$$

The techniques applied here may be generalised to large-n DRG equations of arbitrary complexity, particularly as the $n \to \infty$ limit seems to produce systems of equations with the same principal part. Thus while there may be pedagogical value in drawing a mechanical analogy to large-n DRG equations (Busiello *et al* 1986), existing methods for solving quasi-linear partial differential equations are well suited to deriving invariants, non-linear scaling fields and the global behaviour of RG trajectories. A further discussion of invariants and characteristics of large-n DRG equations will be presented in a subsequent publication.

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